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List of recommened Exercises III

Module 4

901. Write down the HJB equation for the following problem: consider

$$V(0,x)=\sup_{\alpha\in A}\mathbb{E}_{0,x}[e^{-rT}g(X_T^\alpha)],$$

where

$$dX_t^\alpha = rX_t^\alpha dt+\alpha_t X_t^\alpha dB_t,$$

with $X_0=x$.

(This can be interpreted as the pricing equation for an uncertain volatility model with constant interest rate r. The equation is called Black–Scholes–Barenblatt equation and the usual way to present this problem is through a maximisation problem.)

902. Write down the HJB equation for the following problem: consider

$$V(0,x)=\inf_{\alpha\in A}\mathbb{E}_{0,x}[X_T^2],$$

where

$$dX_t^\alpha=\alpha_t dt+dB_t$$

with $X_0=x$.

(The goal in this problem is to bring the state process as close as possible to zero at the terminal time T. However, as defined above, there is no cost of actually controlling the system. This is a simplest example where there is no attainable optimiser.)

903. A Bernoulli equation is an ODE of the form

$$\dot{x}_t+A_tx_t+B_tx_t^\alpha=0,$$

where A,B are are deterministic functions of time and α is a constant. If $\alpha=1$ then the equation is linear and easy to solve. Consider now that $\alpha\neq 1$ and introduce

$$y_t=x_t^{1-\alpha}.$$

Show that y satisfies the linear equation

$$\dot{y}_t+(1-\alpha)A_ty_t+(1-\alpha)B_t=0.$$

(Here the subscript t is not the derivative of t. Rather $A_t=A(t)$.)

904. Sometimes we want to optimise the expected exponential utility criterion:

$$V(t,x) = \mathbb{E}_{t,x}[\exp\{\int_t^T \Psi(s, X_s^\alpha, \alpha_s)ds + \Phi(X_T^\alpha)\}].$$

show that the HJB equation for the expected exponential utility criterion is given by:

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \sup_\alpha\{V(t,x)\Psi(t,x,\alpha) + \quad \mathcal{L}^\alpha V(t,x)\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = e^{\Phi(x)}. \end{cases}$$

1001. Consider the Merton's optimal consumption problem we discussed during F10. Now instead of allowing the investor to be immortal, we let $T < \infty$ be the terminal time of its consumption. Solve the control problem

$$V(t,x) = \sup_{\alpha,c} \mathbb{E}_{t,x}[\int_t^T e^{-\beta t}\Psi(c_t)].$$

where $\beta > 0, \Phi(x) = x^\gamma$, α_t, c_t are as usual, the proportion invested in the risky asset and the consumption rate at time t. Use the ansatz

$$u(t,x) = g(t)x^\gamma.$$

You might need to use the conclusion from Exercise 903.
(Hint. In this problem the terminal condition should be 0 since there's no terminal payoff.)

1002. As we discussed, problems like Exercise 1001 can be generalised to stopping times. Consider

$$\tau = \min\{\inf\{t : X_t = 0\}, T\}$$

instead of T in the previous exercise, the result does not change. Why?

1003. Consider Example 10.1 lecture. Use Ito's formula/Dynkin's lemma to find the optimal control directly for the payoff function $\Phi(x) = x^\gamma$.

1004. Solve the problem

$$\sup_{u\in\mathbb{R}} \mathbb{E}[\int_0^T -u_t^2 dt - X_T^2]$$

where $dX_t = u_t dt + \sigma X_t dB_t$, $X_0 = x, \sigma > 0$. *Hint: Ansatz* $V(t,x) = g(t)x^2$.

1005. Solve the problem

$$\sup_{u\in\mathbb{R}} \mathbb{E}_{t,x}[\int_t^T -u_s^2 ds + X_T^2]$$

where $dX_t = (u_t + \mu X_t)dt + \sigma X_t dB_t$, $X_0 = x, \sigma > 0$. *Hint: Ansatz* $V(t,x) = \exp(f(t) + g(t)x^2)$.

1006. Solve the problem

$$\inf_u \mathbb{E}_{s,x}[\int_0^\infty e^{-\rho(s+t)}(X_t^2 + u_t^2)],$$

where $dX_t = u_t dt + \sigma X_t dB_t$, $X_0 = x, \rho, \sigma > 0$. Hint: consider a two-dimensional process

$$dY_t = \begin{bmatrix} dt \\ dX_t \end{bmatrix} = \begin{bmatrix} 1 \\ u_t \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} dB_t, \quad Y_0 = \begin{bmatrix} s \\ x \end{bmatrix}.$$

Consider ansatz $e^{-\rho s}(ax^2 + b)$ for some constant a, b.

1101. In Example 11.1 we discuss the problem

$$V(x) = \mathbb{E}[e^{-\beta\tau}(X_\tau - c)].$$

Now instead of the linear gain function, let $g(x) = h(x - c)$ and consider now

$$V(x) = \mathbb{E}[e^{-\beta\tau} h(X_\tau - c)].$$

Let $h(x) = x^\gamma, \gamma \in (0, 1)$, write down the free-boundary problem satisfied by V.

1102. In each of the optimal stopping problems below find the supremum V and - if exists - an optimal stopping time τ^*.

(a) $V(x) = \sup_\tau \mathbb{E}[B_\tau^2]$,
(b) $V(x) = \sup_\tau \mathbb{E}[B_\tau^p], p > 0$,
(c) $V(x) = \sup_\tau \mathbb{E}[\exp(-B_\tau^2)]$,
(d) $V(x) = \sup_\tau \mathbb{E}[e^{-\rho(s+\tau)} \cosh B_\tau)], \rho > 0$.

1103. Similarly, back to Example 11.1. Show that when $\mu > \beta$ then $V(x) = \infty$ and τ^* does not exist.

1104. In exercise 1104.a), $V = \infty$ since we don't get penalised for waiting. Now if we add a discounting factor then it becomes

$$V(x) = \sup_\tau \mathbb{E}[e^{-\beta\tau} B_\tau^2]$$

where $\beta > 0$. Solve this problem.

1201. Solve Exercise 11.1.

1202. Solve Exercise 11.1 with utility $h(x) = \log(x)$.

1203. Solve the optimal stopping problem

$$v(x) = \sup_\tau \mathbb{E}_x[e^{-\beta\tau} X_\tau^+].$$

where $\beta > 0$ and $X_t = x + \mu t + B_t$.

1204. Solve the optimal stopping problem

$$v(x) = \sup_{\tau} \mathbb{E}_x[\int_0^{\tau} e^{-\beta t} B_t^2 dt + e^{-\beta\tau} B_\tau^2]$$

where $\beta > 1$. Find V, τ^* and some unsolvable system of b.

1205. Discuss the scenario where $0 < \beta \leq 1$ in Exercise 1204.